Note on Moufang-Noether currents

Eugen Paal

Department of Mathematics
Tallinn University of Technology
Ehitajate tee 5, 19086 Tallinn, Estonia
E-mail: eugen.paal@ttu.ee

Abstract

The derivative Noether currents generated by continuous Moufang tranformations are constructed and their equal-time commutators are found. The corresponding charge algebra turns out to be a birepresentation of the tangent Mal'ltsev algebra of an analytic Moufang loop.

1 Introduction

The Noether currents generated by the Lie transformation groups are well known and widely exploited in modern field theory and theory of elementary particles. Nevertheless, it may happen that group theoretical formalism is too rigid and one has to extend it beyond the Lie groups and algebras. From this point of view it is interesting to elaborate an extension of the group theoretical methods based on Moufang loops as a minimal nonassociative generalization of the group concept. In particular, the Mal'tsev algebra structure of the quantum chiral gauge theory was established in [1, 2].

In this paper, the Noether currents generated by continuous Moufang tranformations (Sec. 2) are constructed. The method is based on the generalized Lie-Cartan theorem (Sec. 3). It turns out that the resulting charge algebra is a birepresentation of the tangent Mal'ltsev algebra of an analytic Moufang loop (Sec. 4). Throughout the paper $\mathrm{i} \doteq \sqrt{-1}$.

2 Moufang loops and Mal'tsev algebras

A Moufang loop [3, 4] is a quasigroup G with the unit element $e \in G$ and the Moufang identity

$$(ag)(ha) = a(gh)a, \qquad a, g, h \in G.$$

Here the multiplication is denoted by juxtaposition. In general, the multiplication need not be associative: $gh \cdot a \neq g \cdot ha$. Inverse element g^{-1} of g is defined

by

$$gg^{-1} = g^{-1}g = e.$$

A Moufang loop G is said [5] to be *analytic* if G is also a real analytic manifold and main operations - multiplication and inversion map $g \mapsto g^{-1}$ - are analytic mappings.

As in the case of the Lie groups, structure constants c^i_{jk} of an analytic Moufang loop are defined by

$$c^i_{jk} \doteq \frac{\partial^2 (ghg^{-1}h^{-1})^i}{\partial g^j \partial h^k}\Big|_{g=h=e} = -c^i_{kj}, \qquad i,j,k = 1,\dots,r \doteq \dim G.$$

Let $T_e(G)$ be the tangent space of G at the unit element $e \in G$. For any $x, y \in T_e(G)$, their (tangent) product $[x,y] \in T_e(G)$ is defined in component form by

$$[x,y]^i \doteq c^i_{jk} x^j y^k = -[y,x]^i, \qquad i = 1,\dots,r.$$

The tangent space $T_e(G)$ being equipped with such an anti-commutative multiplication is called the *tangent algebra* of the analytic Moufang loop G. We shall use notation $\Gamma \doteq \{T_e(G), [\cdot,\cdot]\}$.

The tangent algebra of G need not be a Lie algebra. There may exist such a triple $x, y, z \in T_e(G)$ that does not satisfy the Jacobi identity:

$$J(x,y,z) \doteq [x,[y,z]] + [y,[z,x]] + [z,[x,y]] \neq 0.$$

Instead, for all $x, y, z \in T_e(G)$ one has a more general *Mal'tsev identity* [5]

$$[J(x,y,z), x] = J(x, y, [x,z]).$$

Anti-commutative algebras with this identity are called the *Mal'tsev algebras*. Thus every Lie algebra is a Mal'tsev algebra as well.

3 Birepresentations

Consider a pair (S,T) of the maps $g \mapsto S_g$, $g \mapsto T_g$ of a Moufang loop G into GL_n. The pair (S,T) is called a (linear) *birepresentation* of G if the following conditions hold:

- $S_e = T_e = \mathrm{id}$,
- $T_g S_g S_h = S_{gh} T_g$,
- $S_g T_g T_h = T_{hg} S_g$.

The birepresentation (S,T) is called *associative*, if the following simultaneous relations hold:

$$S_g S_h = S_{gh}, \quad T_g T_h = T_{hg}, \quad S_g T_h = T_h S_g \qquad \forall\, g, h \in G.$$

In general, birepresentations need not be associative even for groups.

4 Structure functions

The *auxiliary functions* of G are defined as

$$v_j^n(g) \doteq \left.\frac{\partial (gh)^n}{\partial h^j}\right|_{h=e}, \qquad j,n = 1,\ldots,r.$$

The *structure functions* $c_{jk}^n(g)$ of G are defined by the *generalized Maurer-Cartan equations*

$$v_j^n(g)\frac{\partial v_k^i(g)}{\partial g^n} - v_k^n(g)\frac{\partial v_j^i(g)}{\partial g^n} \doteq c_{jk}^n(g) v_n^i(g).$$

One can easily check the initial conditions

$$c_{jk}^n(e) = c_{jk}^n.$$

It turns out that the structure functions satisfy the Mal'tsev identity as well.

5 Generalized Lie-Cartan Theorem

The *generators* of a birepresentation (S,T) are defined as

$$S_j \doteq \left.\frac{\partial S_g}{\partial g^j}\right|_{g=e} \quad T_j \doteq \left.\frac{\partial T_g}{\partial g^j}\right|_{g=e} \qquad j = 1,\ldots,r.$$

Define the *derivative generators* of (S,T) as follows:

$$S_j'(g) \doteq T_g S_j T_g^{-1}, \quad T_j'(g) \doteq S_g^{-1} T_j S_g.$$

The *commutator* of matrices is defined by the usual formula $[A,B] \doteq AB - BA$.

Theorem 1 (Generalized Lie-Cartan Theorem, [7]). *The derivative generators of (S,T) satisfy the commutation relations*

$$\begin{aligned}
[S_j'(g), S_k'(g)] &= c_{jk}^n(g) S_n'(g) - 2[S_j'(g), T_k'(g)],\\
[T_j'(g), T_k'(g)] &= -c_{jk}^n(g) T_n'(g) - 2[T_j'(g), S_k'(g)], \quad j,k = 1,\ldots,r.
\end{aligned}$$

Similar commutation relations are known from the theory of alternative algebras [10]. This is due to the fact that the commutator algebras of *alternative* algebras are the Mal'tsev algebras. In a sense, one can also say that the differential of a birepresentation (S,T) of an analytic Moufang loop is a *birepresentation* of its tangent Mal'tsev algebra Γ.

If (S,T) is an associative birepresentation of G we obtain the well known Lie algebra commutation relations (the Lie-Cartan Theorem)

$$[S_j, S_k] - c_{jk}^n S_n = [T_j, T_k] + c_{jk}^n T_n = [S_j, T_k] = 0, \quad j,k = 1,\ldots,r.$$

6 Moufang-Noether currents and ETC

Let us now introduce conventional canonical notations. The coordinates of a space-time point x are denoted by x^α ($\alpha = 0, 1, \ldots, d-1$), where $x^0 = t$ is the time coordinate and x^i are the spatial coordinates denoted concisely as $\mathbf{x} \doteq (x^1, \ldots, x^{d-1})$. The Lagrange density $Ł[u, \partial u]$ is supposed to depend on a system of independent (bosonic or fermionic) fields $u^A(x)$ ($A = 1, \ldots, n$) and their derivatives $\partial_\alpha u^A \doteq u^A_\alpha$. The canonical d-momenta are denoted by

$$p^\alpha_A \doteq \frac{\partial Ł}{\partial u^A_\alpha}.$$

The Moufang-Noether currents are defined in vector (matrix) notations as follows:

$$s^\alpha_j \doteq p^\alpha S'_j(g)u, \qquad t^\alpha_j \doteq p^\alpha T'_j(g)u.$$

and the corresponding Moufang-Noether charges are defined as spatial integrals by

$$\sigma_j(t) \doteq -\,\mathrm{i} \int s^0_j(x) d\mathbf{x}, \qquad \tau_j(t) \doteq -\,\mathrm{i} \int t^0_j(x) d\mathbf{x}.$$

Following the canonical prescription assume that the following equal-time commutators (or anti-commutators when u^A are fermionic fields) hold:

$$\begin{aligned}
[p^0_A(\mathbf{x}, t), u^B(\mathbf{y}, t)] &= -\,\mathrm{i}\, \delta^B_A \delta(\mathbf{x} - \mathbf{y}), \\
[u^A(\mathbf{x}, t), u^B(\mathbf{y}, t)] &= 0, \\
[p^0_A(\mathbf{x}, t), p^0_B(\mathbf{x}, t)] &= 0.
\end{aligned}$$

As a matter of fact, these ETC do not depend on the associativity property of either G nor (S, T). Nonassociativity hides itself in the structure constants of G and in the commutators $[S_j, T_k]$. Due to this, computation of the ETC of the Noether-Moufang charge densities can be carried out in standard way and nonassociativity reveals itself only in the final step when the commutators $[S_j, T_k]$ are required.

First recall that in associative case the Noether charge densities obey the ETC

$$\begin{aligned}
[s^0_j(\mathbf{x}, t), s^0_k(\mathbf{y}, t)] &= \,\mathrm{i}\, c^p_{jk} s^0_p(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y}), \\
[t^0_j(\mathbf{x}, t), t^0_k(\mathbf{y}, t)] &= -\,\mathrm{i}\, c^p_{jk} t^0_p(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y}), \\
[s^0_j(\mathbf{x}, t), t^0_k(\mathbf{y}, t)] &= 0.
\end{aligned}$$

It turns out that for non-associative Moufang transformations these ETC are violated minimally. The Moufang-Noether charge density algebra reads

$$\begin{aligned}
[s^0_j(\mathbf{x}, t), s^0_k(\mathbf{y}, t)] &= \,\mathrm{i}\, c^p_{jk}(g) s^0_p(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y}) - 2[s^0_j(\mathbf{x}, t), t^0_k(\mathbf{y}, t)], \\
[t^0_j(\mathbf{x}, t), t^0_k(\mathbf{y}, t)] &= -\,\mathrm{i}\, c^p_{jk}(g) t^0_p(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y}) - 2[s^0_j(\mathbf{x}, t), t^0_k(\mathbf{y}, t)].
\end{aligned}$$

The ETCs

$$[s_j^0(\mathbf{x},t),t_k^0(\mathbf{y},t)]=[t_j^0(\mathbf{y},t),s_k^0(\mathbf{x},t)]$$

represent associator of an analytic Moufang loop and so may be called the associator as well. Associator of a Moufang loop is not arbitrary but have to fulfil certain constraints [6], the *generalized Lie and Maurer-Cartan equations*. In the present sitiation the constraints can conveniently be listed by closing the above ETC, which in fact means construction of a *finite* dimensional Lie algebra generated by the Moufang-Noether charge densities.

Start by rewriting the Moufang-Noether algebra as follows:

$$[s_j^0(\mathbf{x},t),s_k^0(\mathbf{y},t)]=\mathrm{i}\left[2Y_{jk}^0(x)+\frac{1}{3}c_{jk}^p(g)s_p^0(x)+\frac{2}{3}c_{jk}^p(g)t_p^0(x)\right]\delta(\mathbf{x}-\mathbf{y}). \quad (1)$$

$$[s_j^0(\mathbf{x},t),t_k^0(\mathbf{y},t)]=\mathrm{i}\left[-Y_{jk}^0(x)+\frac{1}{3}c_{jk}^p(g)s_p^0(x)-\frac{1}{3}c_{jk}^p(g)t_p^0(x)\right]\delta(\mathbf{x}-\mathbf{y}). \quad (2)$$

$$[t_j^0(\mathbf{x},t),s_k^0(\mathbf{y},t)]=\mathrm{i}\left[2Y_{jk}^0(x)-\frac{2}{3}c_{jk}^p(g)s_p^0(x)-\frac{1}{3}c_{jk}^p(g)t_p^0(x)\right]\delta(\mathbf{x}-\mathbf{y}). \quad (3)$$

Here (2) can be seen as a definition of the *Yamagutian* $Y_{jk}^0(x)$. The Yamagutian is thus a recapitulation of the associator. It can be shown that

$$Y_{jk}^0(x)+Y_{kj}^0(x)=0, \quad (4)$$

$$c_{jk}^pY_{pl}^0(x)+c_{kl}^pY_{pj}^0(x)+c_{lj}^pY_{pk}^0(x)=0. \quad (5)$$

The constraints (4) trivially descend from the anti-commutativity of the commutator bracketing, but the proof of (5) needs certain effort. Further, it turns out that the following *reductivity* conditions hold:

$$[Y_{jk}^0(\mathbf{x},t),s_n^0(\mathbf{y},t)]=\mathrm{i}\,d_{jkn}^p(g)s_p^0(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}), \quad (6)$$

$$[Y_{jk}^0(\mathbf{x},t),s_n^0(\mathbf{y},t)]=\mathrm{i}\,d_{jkn}^p(g)s_p^0(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}), \quad (7)$$

where the *Yamaguti functions* $d_{jkn}^p(g)$ are defined by

$$6d_{jkn}^p\doteq c_{js}^p(g)c_{kn}^s(g)-c_{ks}^p(g)c_{jn}^s(g)+c_{pn}^p(g)c_{jk}^s(g).$$

Finally, by using the redictivity conditions, one can check that the Yamagutian obeys the Lie algebra

$$[Y_{jk}^0(\mathbf{x},t),Y_{ln}^0(\mathbf{y},t)]=\mathrm{i}\left[d_{jkl}^p(g)Y_{pn}^0(\mathbf{x},t)+d_{jkn}^p(g)Y_{lp}^0(\mathbf{x},t)\right]\delta(\mathbf{x}-\mathbf{y}). \quad (8)$$

When integrating the above ETC (1) - (8) one can finally obtain the

Theorem 2 (Moufang-Noether charge algebra). *The Moufang-Noether charge algebra (σ,τ) is a birepresentation of the Mal'tsev algebra Γ.*

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